

Difference in the number of operators between coupled and uncoupled basis for the general $SU(n)$ Lie algebra

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For the cases of irreducible representation, the complete set of operators necessary to specify uniquely the states. There are two ways of representing the state, using uncoupled and coupled basis. Here we discuss, how the number of operators for the cases of coupled and uncoupled basis changes as well as their difference with the increase of dimension. For higher dimensional groups this number difference changes systematically.

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We are familiar with angular momentum in Quantum Mechanics which is nothing but $SU(2)$ Lie algebra. When we add two angular momenta J_1, J_2 to get the total angular momentum $J (= J_1 + J_2)$ the state of a system can be represented by two different sets of quantum numbers. One way of representing the state is by taking the product of individual angular momentum states and the complete set of quantum numbers are $J_1^2, J_{1z}, J_2^2, J_{2z}$. Then we can consider the coupled basis where the quantum numbers are J^2, J_1^2, J_2^2, J_z which is also a complete set. Now if we generalise this addition of angular momentum to higher $SU(n)$ groups we can formally represent as

$$D(p_1, q_1) \otimes D(p_2, q_2) = \Sigma \oplus \sigma(p, q) D(p, q) \quad (1)$$

where, $D(p_1, q_1)$ and $D(p_2, q_2)$ are two irreducible representation (IR), $\sigma(p, q)$ is an integer and corresponding p_1, q_1, p_2, q_2, p, q are associated with dimensionality of the respective representations.

For $SU(3)$ the complete set of operators necessary to specify uniquely the states of an IR are G^3, F^2, I^2, I_3, Y . So the states of product representation $D(p_1, q_1) \otimes D(p_2, q_2)$ of two IR can be completely specified by the eigenvalues of the 10 linearly independent operators,

$$G^3(1), G^3(2), F^2(1), F^2(2), I^2(1), I^2(2), I_3(1), I_3(2), Y(1), Y(2).$$

If we define the operators of coupled basis (keeping in mind the addition of angular momentum) as

$$O_i = O_i(1) + O_i(2) \quad (2)$$

then we get a set of linearly independent commuting operators

$$G^3, G^3(1), G^3(2), F^2, F^2(1), F^2(2), I^2, I_3, Y.$$

The total number is 9. So this is a non-complete set. We need another operator to make the set complete. Usually the parity operator is taken to make the set complete. Now we can consider higher $SU(n)$ groups and try to see the differences in number of operators when we go from the product basis to coupled basis. To do that we consider the general $SU(n)$ group. The general rule for getting number of operators in a complete set in $SU(n)$ representation is:

there will be $(n - 1)$ number Casimir operators (since rank is $n - 1$), $n - 1$ number weight and number of Casimir operator for each lower group of $SU(n)$ i.e. the number is

$$2(n - 1) + (n - 2) + (n - 3) + \dots + 3 + 2 + 1 = \frac{1}{2}(n + 1)n - 1 \quad (3)$$

In the product basis total number of operators will be twice the number calculated above. So the number is

$$(n + 2)(n - 1). \quad (4)$$

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In the case of coupled basis, we will have $(n - 1)$ casimir operators for each of the two $SU(n)$ groups. Also we will get $(n - 1)$ number of operators analogous to J^2 in case of $SU(2)$ group (in that case the number was $2 - 1 = 1$). Then there will be $(n - 1)$ operators analogous to J_z in the case of $SU(2)$. Finally if we consider the casimir operators of each lower group the total number of operators will be

$$(n - 2) + (n - 3) + + 3 + 2 + 1 = \frac{(n - 1)(n - 2)}{2} \quad (5)$$

So the total number of operators in this case equals to

$$2(n - 1) + (n - 1) + (n - 1) + \frac{(n - 1)(n - 2)}{2} = \frac{1}{2}(n^2 + 5n - 6) \quad (6)$$

Now the difference in the number of operators between the product basis and the coupled basis is (from expressions (4) and (6))

$$(n + 2)(n - 1) - \frac{1}{2}(n^2 + 5n - 6) = \frac{1}{2}(n - 1)(n - 2). \quad (7)$$

Clearly the above number gives correct result for $SU(2)$ and $SU(3)$ which we have already discussed.

Now if we consider the next higher group which is $SU(4)$, in the product basis the complete set of operators is $A^3(1), A^3(2), B^3(1), B^3(2), C^3(1), C^3(2), G^3(1), G^3(2), F^2(1), F^2(2), I^2(1), I^2(2), I_3(1), I_3(2), Y(1), Y(2), Z(1), Z(2)$ and the total number is 18. But in the coupled basis the total number of commuting operators is only 15 which are given below,

$$A^3, A^3(1), A^3(2), B^3, B^3(1), B^3(2), C^3, C^3(1), C^3(2), G^3, F^2, I^2, I_3, Y, Z.$$

Hence the difference in the number of operators is 3, which we can also verify from our general expression (7) by putting $n = 4$.

Similarly we can verify for the cases of higher dimensional group. Thus, if we know the number of complete set of operator in a particular representation then we can calculate number of non complete set of operator in that representation and vice-versa. Here we have introduced a *general rule*, which indicates number difference between complete and non-complete set of operators for any particular representation.

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